

# Open problem on risk-aware planning in the plane

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## Abstract

We consider the problem of planning a collision-free path of a robot in the presence of risk zones. The robot is allowed to travel in these zones but is penalized in a super-linear fashion for consecutive accumulative time spent there. We recently suggested a natural cost function that balances path length and risk-exposure time. When no risk zones exists, our problem resorts to computing minimal-length paths which is known to be computationally hard in the number of dimensions. It is well known that in two-dimensions computing minimal-length paths can be done efficiently. Thus, a natural question we pose is “Is our problem computationally hard or not?” If the problem is hard, we wish to find an approximation algorithm to compute a near-optimal path. If not, then a polynomial-time algorithm should be found.

## 1 Introduction

We are interested in motion-planning problems where an agent has to compute the least-cost path to navigate through *risk zones* while avoiding obstacles. Travelling these regions incurs a penalty which is *super-linear* in the traversal time (see Fig. 1). We call the class of problems *Risk Aware Motion Planning (RAMP)* and use a natural cost function which simultaneously optimizes for paths that are both short and reduce consecutive exposure time in the risk zone.

We are motivated by real-world problems involving *risk*, where continuous exposure is much worse than intermittent exposure. Examples include pursuit-evasion where sneaking in and out of cover is the preferred strategy, and visibility planning where the agent must ensure that an observer or operator is minimally occluded.

In its general form, our problem can be seen as an instance of the *motion-planning problem* (LaValle 2006; Choset et al. 2005) which is known to be PSPACE-Hard (Reif 1979). Thus, in high-dimensional spaces, a natural approach is to follow the *sampling-based paradigm* by computing a discrete graph which is then traversed by a *path-finding* algorithm. Standard path-finding algorithms such as Dijkstra (Dijkstra 1959) and A\* (Hart, Nilsson, and Raphael 1968) cannot be used as optimal plans do not possess optimal substructure. Having said that, we recently suggested ef-

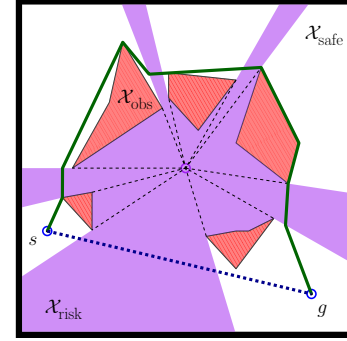


Figure 1: Risk aware motion planning. We need to plan a minimal-cost connecting  $s$  and  $g$  while avoiding obstacles (red). Our cost function penalizes continuous exposure to the risk regions (purple), thus the optimal path (solid green) favours intermittent exposure over the long exposure taken by the shortest path (dotted blue).

ficient path-planning algorithms (Salzman, Hou, and Srinivasa).

When restricting the planning domain to the two-dimensional plane it is not clear whether the problem is computationally hard or not. It is well known that planning for shortest paths in the plane amid polygonal obstacles can be computed in  $O(n \log n)$  time, where  $n$  is the complexity of the obstacles (see (Mitchell 2016) for a survey). When computing shortest paths amid polyhedral obstacles in  $\mathbb{R}^3$ , or in  $\mathbb{R}^2$  when there are constraints on the curvature of the path, the problem becomes NP-Hard (Canny and Reif 1987; Kirkpatrick, Kostitsyna, and Polishchuk 2011). Furthermore, the Weighted Region Shortest Path Problem, which is closely related to our problem (Mitchell and Papadimitriou 1991), is unsolvable in the Algebraic Computation Model over the Rational Numbers (De Carufel et al. 2014). If our problem is computationally hard, as we conjecture, then a reduction, possibly along the lines of (Canny and Reif 1987) should be provided together with an approximation algorithm. For a survey of planning algorithms in low dimensions, see, e.g., (Halperin, Salzman, and Sharir 2016)

## 2 Problem formulation

Let  $\mathcal{P} = \{P_1, \dots, P_m\}$  be a set of simple pairwise interior-disjoint polygons having  $n$  vertices in total. We subdivide  $\mathcal{P}$

Presented in the open problem session of the 12<sup>th</sup> International Workshop on the Algorithmic Foundations of Robotics (WAFR).

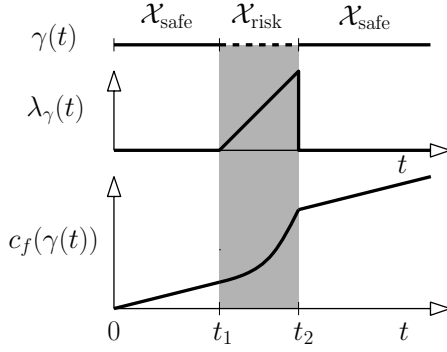


Figure 2: Relation between a trajectory  $\gamma(t)$  (top), recent exposure time  $\lambda_\gamma(t)$  (middle) and cost  $c_f(\gamma(t))$  (bottom) as a function of time. In  $t \in [0, t_1]$ ,  $\gamma$  stays in  $\mathcal{X}_{\text{safe}}$ , hence  $\lambda_\gamma(t) = 0$  and the cost grows linearly with time. At  $t = t_1$ ,  $\gamma$  enters  $\mathcal{X}_{\text{risk}}$ ,  $\lambda_\gamma(t)$  grows linearly and the cost grows super-linearly. At  $t = t_2$ ,  $\gamma$  leaves  $\mathcal{X}_{\text{risk}}$ ,  $\lambda_\gamma(t) = 0$  and the cost returns to growing linearly.

into the disjoint sets  $\mathcal{P}_{\text{obs}}$  and  $\mathcal{P}_{\text{risk}}$  which will be used to define the obstacle region  $\mathcal{X}_{\text{obs}}$  and the risk region  $\mathcal{X}_{\text{risk}}$ , respectively. Roughly speaking, these regions are considered to be open sets. However we do not wish to consider points on the boundary of  $\mathcal{X}_{\text{obs}}$  and  $\mathcal{X}_{\text{risk}}$  as points out of the risk region which are collision free. This is captured by the following definition: The *obstacle* region  $\mathcal{X}_{\text{obs}} = \text{int}(\mathcal{P}_{\text{obs}})$  is the set of all points in the interior of  $\mathcal{P}_{\text{obs}}$ . The *risk* region  $\mathcal{X}_{\text{risk}} = \text{int}(\mathcal{P}_{\text{risk}}) \cup (\mathcal{P}_{\text{obs}} \cap \mathcal{P}_{\text{risk}})$  is the set of all points in the interior of  $\mathcal{P}_{\text{risk}}$  and all points that lie on the border of  $\mathcal{P}_{\text{risk}}$  and  $\mathcal{P}_{\text{obs}}$ . Finally, the *risk-free* region is defined as  $\mathcal{X}_{\text{safe}} = \mathbb{R}^2 \setminus (\mathcal{X}_{\text{risk}} \cup \mathcal{X}_{\text{obs}})$ .

A trajectory  $\gamma : [0, T_\gamma] \rightarrow \mathbb{R}^2$  is a continuous mapping between time and points. We say that  $\gamma$  is *collision free* if  $\forall t \gamma(t) \in \mathcal{X}_{\text{safe}} \cup \mathcal{X}_{\text{risk}}$ . The image of a trajectory is called a path. Given a trajectory  $\gamma$ , and some time  $t \in [0, T_\gamma]$ , let  $t' \leq t$  be the latest time such that  $\gamma(t') \in \mathcal{X}_{\text{safe}}$ . Notice that if  $\gamma(t) \in \mathcal{X}_{\text{safe}}$  then  $t' = t$ . We define the *current exposure time* of  $\gamma$  at  $t$  as  $\lambda_\gamma(t) = t - t'$ . Namely, if  $\gamma(t) \in \mathcal{X}_{\text{risk}}$  then  $\lambda_\gamma(t)$  is the time passed since  $\gamma$  last entered  $\mathcal{X}_{\text{risk}}$ . If  $\gamma(t) \in \mathcal{X}_{\text{safe}}$  then  $\lambda_\gamma(t) = 0$ .

We are now ready to define our cost function. Let  $\gamma$  be a trajectory and  $f(x)$  any function such that  $f(x) = \omega(x)$  and  $f(0) = 1$ . The cost of  $\gamma$ , denoted by  $c_f(\gamma)$  is defined as

$$c_f(\gamma) = \int_{t \in [0, T_\gamma]} f(\lambda_\gamma(t)) |\dot{\gamma}(t)| dt. \quad (1)$$

Eq. 1 penalizes continuous exposure to risk in a super-linear fashion (hence the requirement that  $f(x) = \omega(x)$ ). As  $f(0) = 1$ , the cost of traversing the risk-free region is simply path length. See Fig. 2 for a conceptual visualization of the current exposure time and our cost function.

Equipped with our cost function we can formally state the risk-aware motion-planning problem:

**Planar Risk-aware motion-planning problem (pRAMP)**  
Given the tuple  $(\mathcal{X}_{\text{safe}}, \mathcal{X}_{\text{risk}}, \mathcal{X}_{\text{obs}}, s, g, f)$ , where  $s, g \in \mathcal{X}_{\text{safe}}$  are start and goal points, compute  $\arg \min_{\gamma \in \Gamma} c_f(\gamma)$

with  $\Gamma$  the set of all collision-free trajectories connecting  $s$  and  $g$

We defined our problem to be as general as possible. However, to simplify the discussion, we assume that the robot is moving in constant speed and we use  $f(x) = e^x$ . Thus, we can re-write Eq. 1 as

$$c(\gamma) = \int_{t \in [0, T_\gamma]} e^{\lambda_\gamma(t)} dt. \quad (2)$$

Using the constant-speed assumption, we can use the terms duration of a trajectory and path length interchangeably (here we measure path length as the Euclidean distance). Further exploiting this assumption and by a slight abuse of notation we can also use Eq. 2 to define the cost of a path (and not of a trajectory). For different properties of this cost function, see (Salzman, Hou, and Srinivasa ).

### 3 Discussion and open questions

#### 3.1 Hardness

When considering the complexity of a planning problem, one needs to consider both the algebraic complexity and the combinatorial complexity. If we use the Algebraic Computation Model over the Rational Numbers (ACM $\mathbb{Q}$ ), then we conjecture that the problem is unsolvable. A proof may follow the lines taken in (De Carufel et al. 2014) for the Weighted Region Shortest Path Problem.

Assessing the combinatorial complexity of our problem, defined analogously to the number of “edge sequences” is not as straightforward. Several hardness results for planning problems use reductions from 4CNF-satisfiability (Asano, Kirkpatrick, and Yap 2003; Kirkpatrick, Kostitsyna, and Polishchuk 2011). The proofs use the idea of “path encoding” which involves constructing an environment that admits an exponential number of distinct shortest paths between  $s$  and  $t$ . Each path is associated with a truth assignment of a given formula  $\Phi$ . Then, the environment is augmented with additional obstacles that block every path whose associated truth assignment does not satisfy the formula  $\Phi$ . The underlying problem with using this approach is that in the plane it depends heavily on the fact that a minimal-cost paths can self-intersect, which is not the case in our setting.

#### 3.2 Approximation algorithm

Assuming that the problem is computationally hard, we seek an approximation algorithm such that given some  $\varepsilon$  returns a path whose cost is at most  $1 + \varepsilon$  the cost of the optimal path in time polynomial in  $n$  and  $\varepsilon$ . A natural approach would be to sample densely along the boundary of  $\mathcal{X}_{\text{risk}}$  and compute the visibility graph defined over the sampled points and the vertices in  $\mathcal{P}$ . A minimal-cost path may then be computed in polynomial time (Salzman, Hou, and Srinivasa ). However, the running time of this algorithm also depends on the *length* of the edges of polygons in  $\mathcal{P}$  (see similar approach and analysis in (Wein, van den Berg, and Halperin 2008)).

We believe that a possible approach would be to sample the boundary of  $\mathcal{X}_{\text{risk}}$  more carefully, similar to (Agarwal, Fox, and Salzman 2016).

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